

## Some simple solutions for heat-induced tropical circulation

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### SUMMARY

A simple analytic model is constructed to elucidate some basic features of the response of the tropical atmosphere to diabatic heating. In particular, there is considerable east–west asymmetry which can be illustrated by solutions for heating concentrated in an area of finite extent. This is of more than academic interest because heating in practice tends to be concentrated in specific areas. For instance, a model with heating symmetric about the equator at Indonesian longitudes produces low-level easterly flow over the Pacific through propagation of Kelvin waves into the region. It also produces low-level westerly inflow over the Indian Ocean (but in a smaller region) because planetary waves propagate there. In the heating region itself the low-level flow is away from the equator as required by the vorticity equation. The return flow toward the equator is farther west because of planetary wave propagation, and so cyclonic flow is obtained around lows which form on the western margins of the heating zone. Another model solution with the heating displaced north of the equator provides a flow similar to the monsoon circulation of July and a simple model solution can also be found for heating concentrated along an inter-tropical convergence line.

### 1. INTRODUCTION

The first theory of tropical circulation can be attributed to Halley (1686), who proposed that heating in the tropics caused air to become ‘less ponderous’ and hence to rise. This explained the equatorward component of the trade winds (which Halley stated would be accompanied by a reverse flow aloft), but not the easterly component. That was explained later by Hadley (1735) in terms of the tendency of the equatorward flowing air to conserve its angular momentum about the earth’s axis. The zonally averaged seasonal pictures of the meridional circulation (e.g. Newell *et al.* 1974) show the motion envisaged by Halley, although the picture is not symmetric about the equator, the sinking part of the circulation being nearly all in the winter hemisphere.

However, the tropical circulation is by no means zonally symmetric. In particular, Bjerknes (1969) drew attention to the Walker circulation in the Pacific Ocean. This is a circulation in the plane of the equator with rising air over the warm zone in the West Pacific and descending motion over the cold ocean of the East Pacific. Variations in the Walker circulation seem to be an important feature of the Southern Oscillation (Julian and Chervin 1978).

Much of the heating in the tropics is found to be over the three continental areas of the tropics, i.e. Africa, South America and the Indonesian region (see e.g. Ramage 1968 and the discussion in Krueger and Winston 1974), each of which is relatively small in extent. That causes speculation on the result of heating a limited area centred on or near the equator at a particular longitude.

If the heating is applied to a resting atmosphere, and is small enough to use linear theory, the response can be modelled in terms of equatorially trapped waves. If the heating were switched on at some initial time, Kelvin waves would carry the information rapidly

eastward, thereby creating easterly trade winds in that region, the trades providing inflow to the region of heating and so providing a Walker-type circulation with rising over the source region and sinking to the east. If dissipative processes are added, a steady state can be reached with the trade winds gradually dying out with distance from the source region. Thus trade winds over the Pacific are generated in this model by the heating over Indonesia.

In addition, the switching on of the heating would generate a planetary wave which would carry information westwards into the Indian Ocean. The fastest planetary wave, however, travels at a third of the Kelvin wave speed, so this effect would be expected to penetrate only one-third as far as the Kelvin wave. This planetary wave response could explain the surface westerlies in the Indian Ocean (Lettau 1973), as being a result of the heating over the Indonesian region. Such qualitative results are sufficiently realistic to suggest it would be useful to produce a simple mathematical model on the above lines, and this is done in subsequent sections. The model is designed to show how the response to heating varies in the horizontal. Although emphasis is given to problems with forcing in limited regions (which nicely illustrate east-west asymmetries in the response), the model can also be applied to more general forms of forcing.

## 2. THE MODEL

The aim is to study the response of tropical atmosphere to a given distribution of heating, using as simple a model as possible. It is natural, therefore, to use the *linear* theory for small perturbations a resting atmosphere, i.e. the heating rate will be assumed small enough for linear theory to apply. Other effects may well play a significant role in practice but will not be considered here.

The unperturbed state, therefore, consists of an atmosphere at rest with properties a function of height  $z$  only. In the absence of dissipative processes and forcing, an effective means of studying motions with large horizontal scale (i.e. large compared with the vertical scale) has been separation of the solution into a height-dependent part and a part which depends on the horizontal coordinates and time. The idea was already implied in the discussion by Laplace (1778/9) of thermal oscillations in the atmosphere where he effectively stated that there exists a mode which satisfies his tidal equations with an equivalent depth equal to the scale height. (This is now called the Lamb mode.) Taylor (1936) showed how the technique can be applied to a compressible atmosphere with undisturbed temperature an arbitrary function of height. Details are discussed, e.g. by Holton (1975). In the case of an isothermal atmosphere, for instance, there is a continuous spectrum of modes for each of which the square root of density times the vertical displacement varies sinusoidally with altitude and one additional mode (the Lamb mode) for which the vertical displacement is zero but the pressure perturbation decays exponentially with height. For each of these modes, variations with horizontal position and with time are governed by the 'shallow water' equations but with a different 'equivalent depth' of water for each mode.

A technique for dealing with forced problems is to expand the forcing function in terms of normal modes. This method has been applied, for instance, by Lighthill (1969) to an oceanographic problem. The equivalent in the present problem is to express the diabatic heating rate as a Fourier-type integral over the complete set of modes. Important contributions to the solution will be expected to come from modes with inverse vertical wavenumber  $m^{-1}$  of the same order as the scale of the forcing function. Since, in the tropical atmosphere, the diabatic heating tends on average (see e.g. Hantel and Baader 1978) to be a maximum at 5 km, this can be taken as a representative value for  $m^{-1}$ . The wave speed for such a mode

in the absence of rotation is about  $60 \text{ m s}^{-1}$  (it is approximately given by  $N/m$  where  $N$  is a mean value for the buoyancy frequency, compressibility effects being minor at this scale) corresponding to an equivalent depth of about 400 m.

The above remarks are intended merely to put the present calculation in perspective, for attention here is concentrated on structure in the *horizontal* and solutions will be found for one mode only. Since the solutions are all for forcing in a limited region, solutions for different modes with different (but similar magnitude) equivalent depths would have similar structure but with slightly different scales.

There are other approaches which lead to the same shallow-water equations, e.g. a two-layer atmospheric model with small perturbations about a rest state would give the same set. In this model, heating is equivalent to increasing the amount of high potential temperature fluid, i.e. transferring mass from the lower layer to the upper layer (cf. Gill 1979; Gill *et al.* 1979). Another approach is to consider an incompressible atmosphere with constant buoyancy frequency  $N$  and with a rigid lid at height  $z = D$ . The gravest mode in this case has pressure perturbations and horizontal velocity components which vary with height as  $\cos(\pi z/D)$ , a vertical velocity which varies like  $\sin(\pi z/D)$ , and an equivalent depth  $H$  given by

$$c = (gH)^{\frac{1}{2}} = ND/\pi, \quad (2.1)$$

where  $g$  is the acceleration due to gravity and  $c$  is a separation constant which is equal to the speed of long waves in the absence of rotation. If the diabatic heating rate is chosen to vary like  $\sin(\pi z/D)$ , then only the gravest mode will be stimulated and hence the shallow-water equations for a single mode describe the complete solution. It is convenient when pictorial representations of the vertical structure are required to use this mode. It should be remembered, however, that the solution has wider significance as indicated above.

The problem has now been reduced to solving the forced shallow-water equations in the tropics. Because the motion is confined to the tropics, it is convenient to use the equatorial beta-plane approximation in which the Coriolis parameter is approximated as  $\beta$  times distance northwards from the equator. It is also convenient to write the equations in a non-dimensional form (Gill and Clarke 1974), using as length scale the equatorial Rossby radius  $(c/2\beta)^{\frac{1}{2}}$ , which is about  $10^\circ$  of latitude when the equivalent depth is 400 m, and the time scale  $(2\beta c)^{-\frac{1}{2}}$ , which is about one-quarter of a day. Justification of the beta-approximation comes from the small size of the Rossby radius compared with the  $90^\circ$  span from equator to poles. The equations have the form (cf. Matsuno 1966):

$$\frac{\partial u}{\partial t} - \frac{1}{2}yv = -\frac{\partial p}{\partial x}, \quad (2.2)$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}yu = -\frac{\partial p}{\partial y}, \quad (2.3)$$

$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -Q. \quad (2.4)$$

In these equations  $(x, y)$  is non-dimensional distance with  $x$  eastwards and  $y$  measured northwards from the equator,  $(u, v)$  is proportional to horizontal velocity and  $p$  is proportional to the pressure perturbation.  $Q$  is proportional to the heating rate, the signs being such that if  $Q$  is positive (positive heating), the signs of  $u, v, p$  correspond to those at the surface. Equations (2.2) and (2.3) are the momentum equations, while (2.4) is the modal version of the continuity equation, the vertical velocity being proportional to

$$w = \frac{\partial p}{\partial t} + Q, \quad (2.5)$$

the latter being derived from the buoyancy equation.

In order to study the response to steady forcing, dissipative processes must be included somehow. The most convenient is the so-called 'Rayleigh friction' and 'Newtonian cooling' forms which replace the operator  $\partial/\partial t$  by  $\partial/\partial t + \varepsilon$ . The mathematics is simplest when the epsilon for friction is the same as the one for cooling, so the steady-state versions of Eqs. (2.2) to (2.5) are

$$\varepsilon u - \frac{1}{2} y v = -\frac{\partial p}{\partial x}, \quad (2.6)$$

$$\varepsilon v + \frac{1}{2} y u = -\frac{\partial p}{\partial y}, \quad (2.7)$$

$$\varepsilon p + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -Q, \quad (2.8)$$

$$w = \varepsilon p + Q. \quad (2.9)$$

This model was used by Matsuno (1966).

The above set of equations can be reduced to a single equation for  $v$  namely

$$\varepsilon^3 v + \frac{1}{4} \varepsilon y^2 v - \varepsilon \frac{\partial^2 v}{\partial x^2} - \varepsilon \frac{\partial^2 v}{\partial y^2} - \frac{1}{2} \frac{\partial v}{\partial x} = \varepsilon \frac{\partial Q}{\partial y} - \frac{1}{2} y \frac{\partial Q}{\partial x}. \quad (2.10)$$

(Note: McCreary (1980) has shown, with suitable assumptions, this method can be extended to include all vertical modes. Vertical eddy transfer can be allowed for by making  $\varepsilon$  a function of  $m$  and hence of  $H$ .)

One further approximation will now be made. The forcing will be chosen to have a  $y$ -scale of order unity and  $\varepsilon$  will be assumed small. Hence the term  $\varepsilon^3 v$  at the beginning of Eq. (2.10) can be neglected. Secondly, if the forcing has east-west wavenumber  $k$ , the term  $\varepsilon \partial^2 v / \partial x^2$  is small compared with the term  $\frac{1}{2} \partial v / \partial x$  provided

$$2\varepsilon k \ll 1. \quad (2.11)$$

This will be assumed to be true also, i.e. that the forcing has east-west scale large compared with  $2\varepsilon$ . This is no real restriction because  $\varepsilon$  is assumed to be small. These approximations are equivalent to neglecting the term  $\varepsilon v$  in Eq. (2.7), which now becomes

$$\frac{1}{2} y u = -\frac{\partial p}{\partial y} \quad (2.12)$$

i.e. the eastward flow is in geostrophic balance with the pressure gradient. This is also equivalent to making the 'long wave' approximation in the transient problem (cf. Gill and Clarke 1974).

### 3. METHOD OF SOLUTION

In order to solve the three Eqs. (2.6), (2.8) and (2.12), it is convenient first to introduce two new variables  $q$  and  $r$  to replace  $p$  and  $u$ . These are defined (Gill 1975) by

$$q = p + u \quad (3.1)$$

$$r = p - u. \quad (3.2)$$

The sum and difference of (2.8) and (2.6) then yield

$$\varepsilon q + \frac{\partial q}{\partial x} + \frac{\partial v}{\partial y} - \frac{1}{2} y v = -Q \quad (3.3)$$

$$\varepsilon r - \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} y v = -Q \quad (3.4)$$

while Eq. (2.12) can be rewritten in the form

$$\frac{\partial q}{\partial y} + \frac{1}{2} y q + \frac{\partial r}{\partial y} - \frac{1}{2} y r = 0. \quad (3.5)$$

The free solutions of (3.3), (3.4) and (3.5) have the form of parabolic cylinder functions  $D_n(y)$  (Abramowitz and Stegun 1965, Ch. 19) and solutions of the forced problem can be found by expanding the variables  $q$ ,  $r$ ,  $v$  and  $Q$  in terms of these functions, namely

$$q = \sum_{n=0}^{\infty} q_n(x) D_n(y) \quad (3.6)$$

etc. These have the properties

$$\frac{dD_n}{dy} + \frac{1}{2} y D_n = n D_{n-1}, \quad (3.7)$$

$$\frac{dD_n}{dy} - \frac{1}{2} y D_n = -D_{n+1} \quad (3.8)$$

So substituting the expressions of the form (3.6) into (3.3), (3.4) and (3.5) gives

$$\left. \begin{aligned} \varepsilon q_0 + \frac{dq_0}{dx} &= -Q_0 \\ \varepsilon q_{n+1} + \frac{dq_{n+1}}{dx} - v_n &= -Q_{n+1} \quad n \geq 0 \end{aligned} \right\} \quad (3.9)$$

$$\varepsilon r_{n-1} - \frac{dr_{n-1}}{dx} + n v_n = -Q_{n-1} \quad n \geq 1, \quad (3.10)$$

$$\left. \begin{aligned} q_1 &= 0 \\ r_{n-1} &= (n+1)q_{n+1} \quad n \geq 1 \end{aligned} \right\} \quad (3.11)$$

In the following sections solutions will be found for two special cases where the forcing has a particularly simple form. One is the case where the heating rate  $Q$  is symmetric about the equator and has the form

$$Q(x, y) = F(x) D_0(y) = F(x) \exp(-\frac{1}{4} y^2). \quad (3.12)$$

The second has heating antisymmetric about the equator and of the form

$$Q(x, y) = F(x) D_1(y) = F(x) y \exp(-\frac{1}{4} y^2). \quad (3.13)$$

The advantage of these forms is that the response only involves parabolic cylinder functions up to order 3, and these are given by

$$D_0, D_1, D_2, D_3 = (1, y, y^2 - 1, y^3 - 3y) \exp(-\frac{1}{4} y^2). \quad (3.14)$$

The forcing will be assumed localized in the neighbourhood of  $x = 0$  and to have the form

$$F(x) = \begin{cases} \cos kx & |x| < L \\ 0 & |x| > L \end{cases} \quad (3.15)$$

where

$$k = \pi/2L. \quad (3.16)$$

This is the case where the heating rate has the form (3.12), that is, the only non-zero coefficient  $Q_n$  corresponds to  $n = 0$  and is given by

For this form of forcing, there are two parts of the response. The first part involves  $q_0$  only, and this coefficient satisfies (3.9) with  $n = 0$ . This part represents a *Kelvin wave* which is damped out as it progresses *eastwards*. The wave moves at unit speed with decay rate  $\varepsilon$ , so has spatial decay rate  $\varepsilon$  as well. Since no information is carried westwards, the solution is zero for  $x < -L$  and so the solution of (3.9) is given by

Consider now what this Kelvin-wave part of the solution looks like. Using (3.1), (3.2), (3.9) and (2.5) it follows that this part of the solution is given by

If the forcing region is taken to represent the Indonesian area, this solution represents the *Walker circulation* over the Pacific with easterly trades flowing parallel to the equator into the forcing region, rising there and then flowing eastward aloft. A corresponding pressure trough is found on the equator. Pressure decreases towards the west and the flow along the equator is down the pressure gradient.

The second part of the forcing corresponds to putting  $n = 1$  in (3.9), (3.11) and (3.10) which then give

This corresponds to the  $n = 1$  *long planetary wave* which propagates *westwards* at speed  $1/3$  so giving a spatial decay rate of  $3\varepsilon$ . As no information is carried eastwards, the solution of (4.6) is given by

with  $n = 1$ .

The detailed solution for the pressure and velocity components follows from (3.1), (3.2), (4.4), (4.5), (4.6), (2.5) and (3.14), namely

$$\left. \begin{aligned} p &= \frac{1}{2}q_2(x)(1+y^2)\exp(-\frac{1}{4}y^2) \\ u &= \frac{1}{2}q_2(x)(y^2-3)\exp(-\frac{1}{4}y^2) \\ v &= \{F(x)+4\epsilon q_2(x)\}y\exp(-\frac{1}{4}y^2) \\ w &= \frac{1}{2}\{F(x)+\epsilon q_2(x)(1+y^2)\}\exp(-\frac{1}{4}y^2) \end{aligned} \right\} \quad (4.8)$$

Consider the properties of this solution, beginning with the region  $x < -L$  to the west of the forcing zone. Here  $q_2$  is negative, so there is a net eastward flow in the lower layer given by

$$\int_{-\infty}^{\infty} u dy = -\pi^{\frac{1}{2}}q_2(x). \quad (4.9)$$

The winds are westerly along the equator and fall to zero at  $y = \sqrt{3}$ , with weak easterlies at higher latitudes. The air is descending and directed equatorwards throughout the entire region and there is a trough of low pressure along the equator. The pressure decreases to the east and flow along the equator is down the pressure gradient.

It remains now to consider the flow in the forcing region  $|x| < L$ , which requires summing the contributions from the part (4.3) and the part (4.8). The  $p$  and  $u$  fields are much as expected, but the meridional flow shows a feature which is somewhat surprising. This feature is most marked in the limit as  $\epsilon$  tends to zero when (4.8) gives

$$(v, w) \rightarrow (y, 1)F(x)\exp(-\frac{1}{4}y^2). \quad (4.10)$$

Thus the flow in the heating region is upwards but *away from* and not towards the heat source. The explanation comes from the vorticity equation obtained by taking the curl of (2.6) and (2.7). This gives in the limit as  $\epsilon \rightarrow 0$

$$y\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v = 0, \quad (4.11)$$

i.e. the divergence is balanced by the ' $\beta v$ ' term. Substitution from (2.8) with  $\epsilon = 0$  gives

$$v = yQ, \quad (4.12)$$

which is the equivalent of the Sverdrup equation in oceanography. (Note that this gives the opposite sign to that obtained by putting  $u = 0$  in (2.8).)

Now the explanation follows easily. Heating, in the limit as  $\epsilon \rightarrow 0$ , causes upward motion by (2.9). This upward motion causes vortex lines in the lower layer to be stretched, and so increase their absolute vorticity (cyclonic positive). In the zero-epsilon limit, however, particles must achieve this by changing to a latitude where their vorticity is the same as that of the background. (This is required by (4.14), i.e. by moving polewards.) Thus heating causes poleward motion in the lower layer and equatorward motion aloft. If  $\epsilon$  is allowed to become non-zero, the rotational constraint is less strong and so the region where heating causes poleward motion becomes smaller in extent, but (4.7) and (4.8) show that  $v$  is always positive near  $x = L$ .

Despite the above 'peculiarity' of the forcing region, the zonally integrated flow has the expected form of a double Hadley cell with symmetry about the equator. The integrated flow is most easily obtained by integrating (3.9), (4.4), (4.5) and (4.6) direct, which gives

$$\int_{-\infty}^{\infty} (q_0, q_2, r_0, v_1) dx = \left(-1, -\frac{1}{3}, -\frac{2}{3}, -\frac{\epsilon}{3}\right) I/\epsilon \quad (4.13)$$

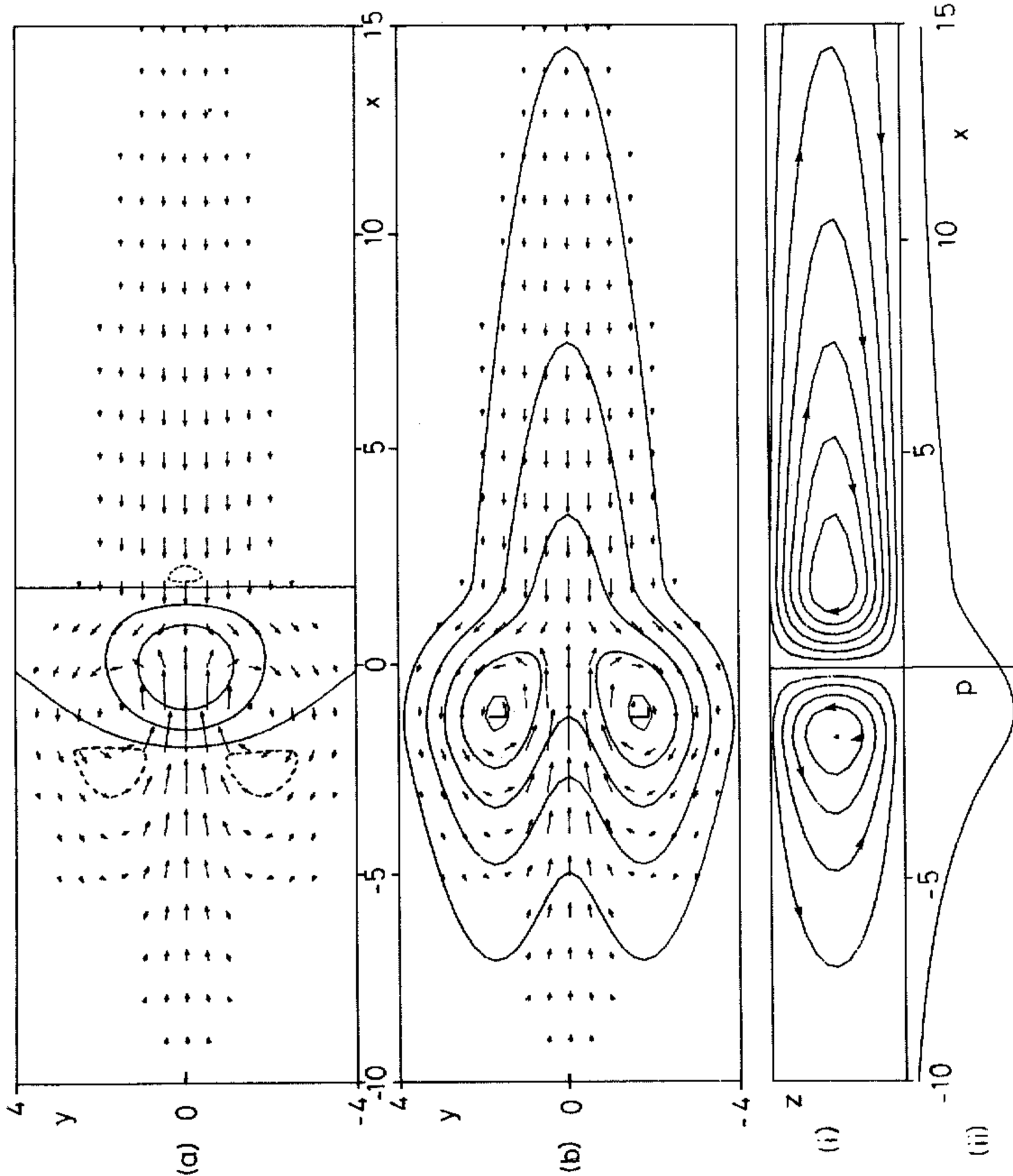
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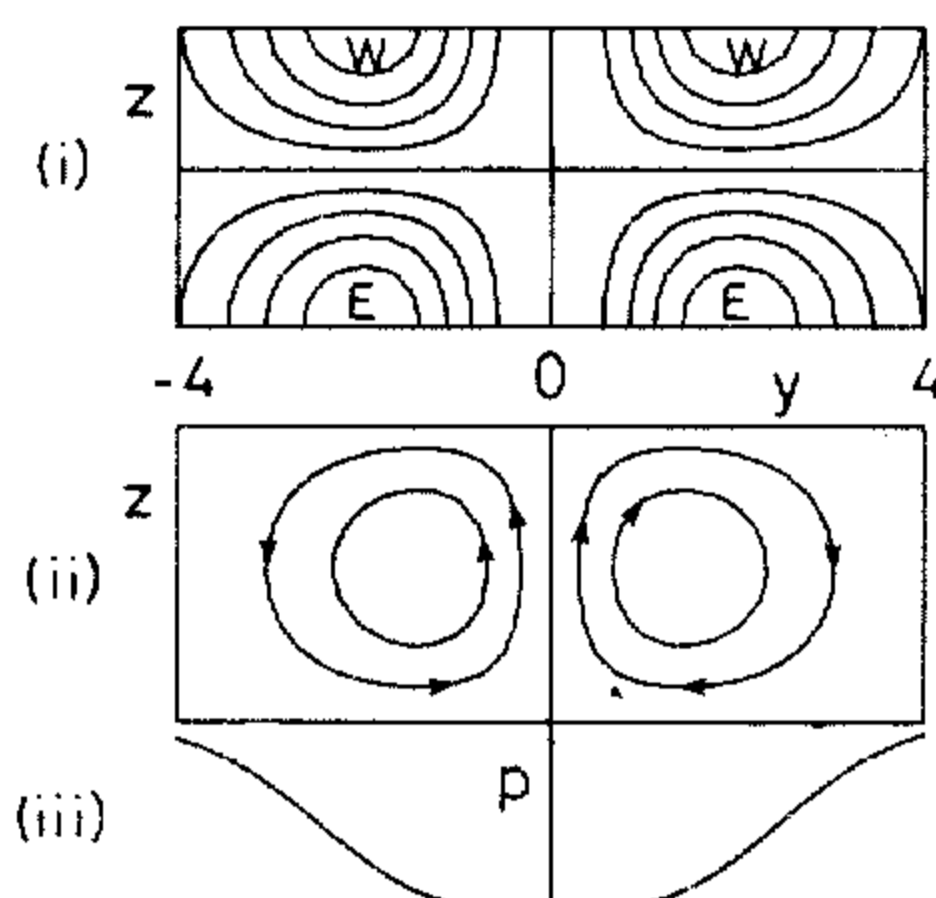
Figure 1. Solution for heating symmetric about the equator in the region  $|x| < 2$  for decay factor  $\varepsilon = 0.1$ .

(a) Contours of vertical velocity  $w$  (solid contours are 0, 0.3, 0.6, broken contour is  $-0.1$ ) superimposed on the velocity field for the lower layer. The field is dominated by the upward motion in the heating region where it has approximately the same shape as the heating function. Elsewhere there is subsidence with the same pattern as the pressure field.

(b) Contours of perturbation pressure  $p$  (contour interval 0.3) which is everywhere negative. There is a trough at the equator in the easterly regime to the east of the forcing region. On the other hand, the pressure in the westerlies to the west of the forcing region, though depressed, is high relative to its value off the equator. Two cyclones are found on the north-west and south-west flanks of the forcing region.

(c) The meridionally integrated flow showing (i) stream function contours, and (ii) perturbation pressure. Note the rising motion in the heating region (where there is a trough) and subsidence elsewhere. The circulation in the right-hand (Walker) cell is five times that in each of the Hadley cells shown in (c).





(d) The zonally integrated solution showing (i) contours of zonal velocity (E marks the core of the easterly flow and W of the westerly flow aloft), (ii) stream function contours, and (iii) perturbation pressure.

$$I = \int_{-\infty}^{\infty} Q_0 dx = \int_{-L}^L F dx = 2/k = 4L/\pi. \quad (4.14)$$

The integrated pressure and velocity fields follow from (3.1), (3.2), (2.9), (3.6) and (3.14) which give

$$\int_{-\infty}^{\infty} (p, u, v, w) dx = \left( -\frac{4+y^2}{6\varepsilon}, \frac{-y^2}{6\varepsilon}, \frac{-y}{3}, \frac{2-y^2}{6} \right) I \exp(-\frac{1}{4}y^2). \quad (4.15)$$

Thus there is an equatorial trough, easterly jets at the surface on either side of the equator and the expected Hadley circulation. A sketch is shown in Fig. 1 using for convenience the vertical structure associated with the rigid lid. The horizontal pattern obtained in Fig. 1(b) shows some similarity with the solution obtained by Matsuno (1966) for forcing which is periodic in  $x$ . However, the east-west asymmetry is brought out much more clearly with the present example.

It is also of interest to calculate the meridional integrals of the flow using the integrals

$$\int_{-\infty}^{\infty} (1, y, y^2) \exp(-\frac{1}{4}y^2) dy = (2, 0, 4)\pi^{\frac{1}{2}}. \quad (4.16)$$

Then integrating (4.3) plus (4.8) gives

$$\left. \begin{aligned} \int_{-\infty}^{\infty} p dy &= \{q_0(x) + 3q_2(x)\}\pi^{\frac{1}{2}} \\ \int_{-\infty}^{\infty} u dy &= \{q_0(x) - q_2(x)\}\pi^{\frac{1}{2}} \\ \int_{-\infty}^{\infty} w dy &= \{\varepsilon q_0(x) + 3\varepsilon q_2(x) + 2F(x)\}\pi^{\frac{1}{2}} \end{aligned} \right\}. \quad (4.17)$$

This solution is also shown in Fig. 1 and shows inflow into the heating region (where there is a trough) near the surface and outflow aloft. The zone of easterlies to the east of the forcing region is more extensive than the zone of westerlies to the west because of the three-times greater speed of the Kelvin wave relative to the long  $n = 1$  planetary wave. The circulation in the 'Walker' cell is about five times that in each of the Hadley cells.

The flow pattern in the lower troposphere for the case  $\varepsilon = 0.1$ ,  $L = 2$  is shown in Fig. 1 and shows the general characteristics already deduced above. To the east of the



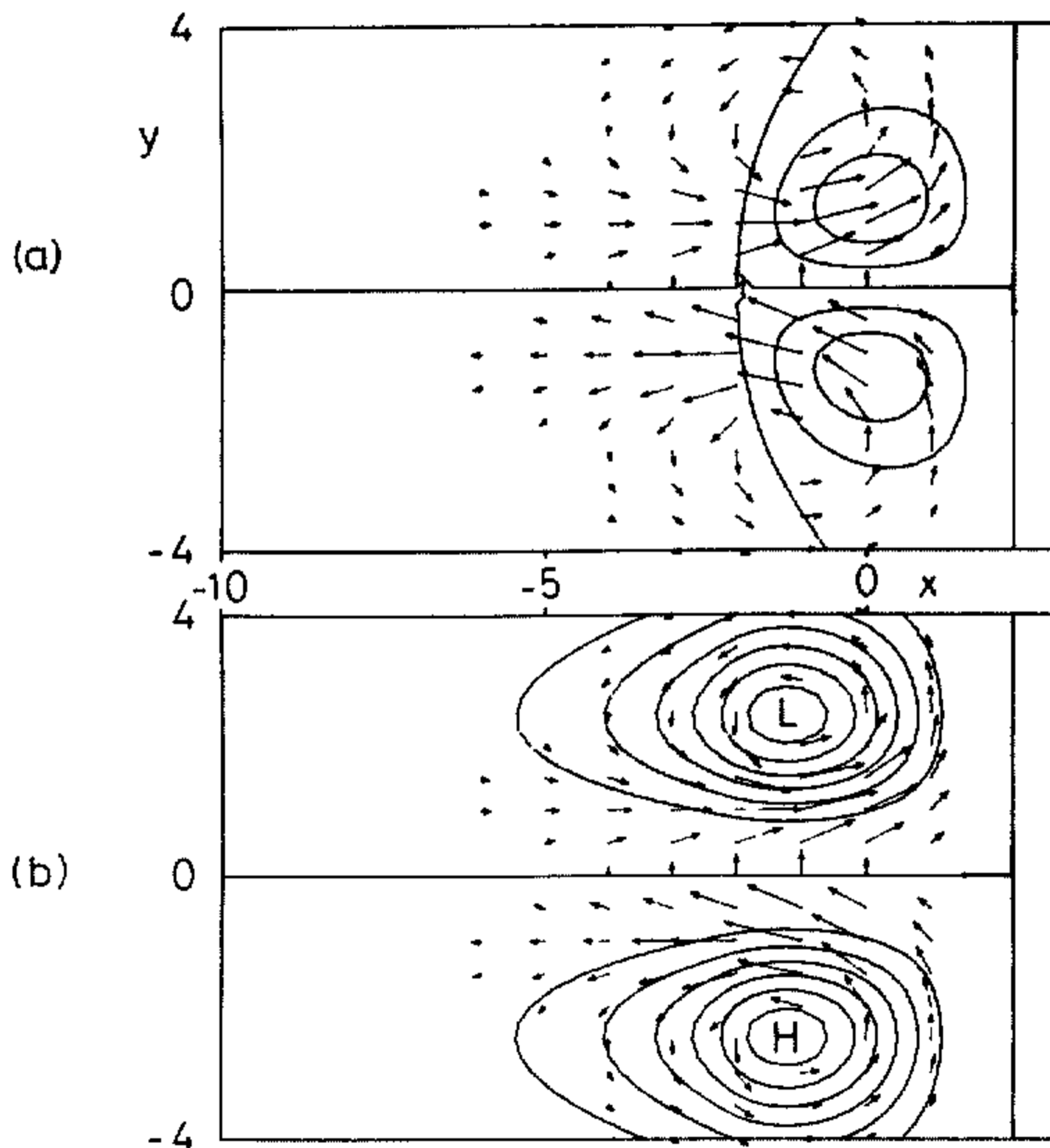
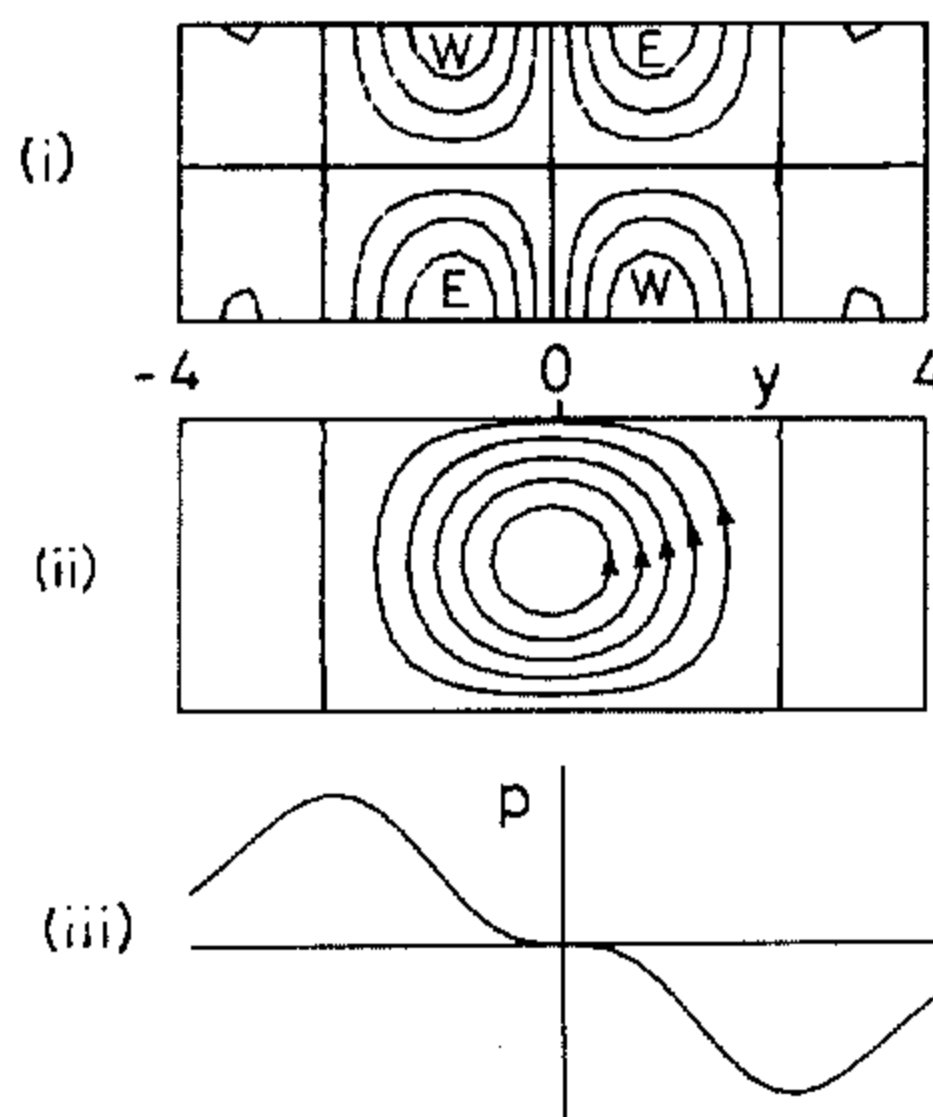


Figure 2. Solution for heating antisymmetric about the equator in the region  $|x| < 2$  for decay factor  $\epsilon = 0.1$ .  
 (a) Contours of vertical velocity  $w$  (contour interval 0.3) superimposed on the velocity field for the lower layer. The field is dominated by the motion in the heating region where it is approximately the same shape as the heating function (positive in northern hemisphere). Outside the forcing region, the pattern is the same as for the pressure field with subsidence in the northern hemisphere and weak upward motion in the southern hemisphere. There is no motion for  $x > 2$ .  
 (b) Contours of perturbation pressure  $p$  (contour interval 0.3) which is positive in the north (where there is an anticyclone) and negative south of the equator (where there is a cyclone). The flow has the expected sense of rotation around the pressure centres and flows down the pressure gradient where it crosses the equator. All fields are zero for  $x > 2$ .



(c) The zonally integrated solution showing (i) contours of eastward velocity, (ii) stream function contours, and (iii) perturbation pressure.

gates that way. To the west there is a region of westerly inflow into the heating region in the heating hemisphere, but this is somewhat limited in extent because the spatial decay rate of the long  $n = 2$  planetary wave is  $5\epsilon$ . In the heating region there is still a tendency for poleward flow because (4.12) still applies in the limit as  $\epsilon$  tends to zero. This implies a cyclone centred westwards of the centre of the heating region, and this shows up prominently in the Figure.

The zonally integrated motion is obtained by integrating (5.5) and (5.6), the result being

$$\int_{-\infty}^{\infty} (p, u, v, w) dx = \left( -\frac{y^3}{10\epsilon}, \frac{6y-y^3}{10\epsilon}, \frac{6-y^2}{5}, \frac{10y-y^3}{10} \right) I \exp(-\frac{1}{4}y^2). \quad (5.7)$$

This is sketched in Fig. 2. There is a dominant Hadley cell spanning the equator with rising in the heating hemisphere (where there is a trough). There is low-level westerly flow where it is also polewards, as one would expect from the tendency to conserve angular momentum.

## 6. FURTHER SOLUTIONS

The same technique can be applied with forcing corresponding to other coefficients  $Q_n$  and the solution for general forcing can be obtained by adding contributions from the

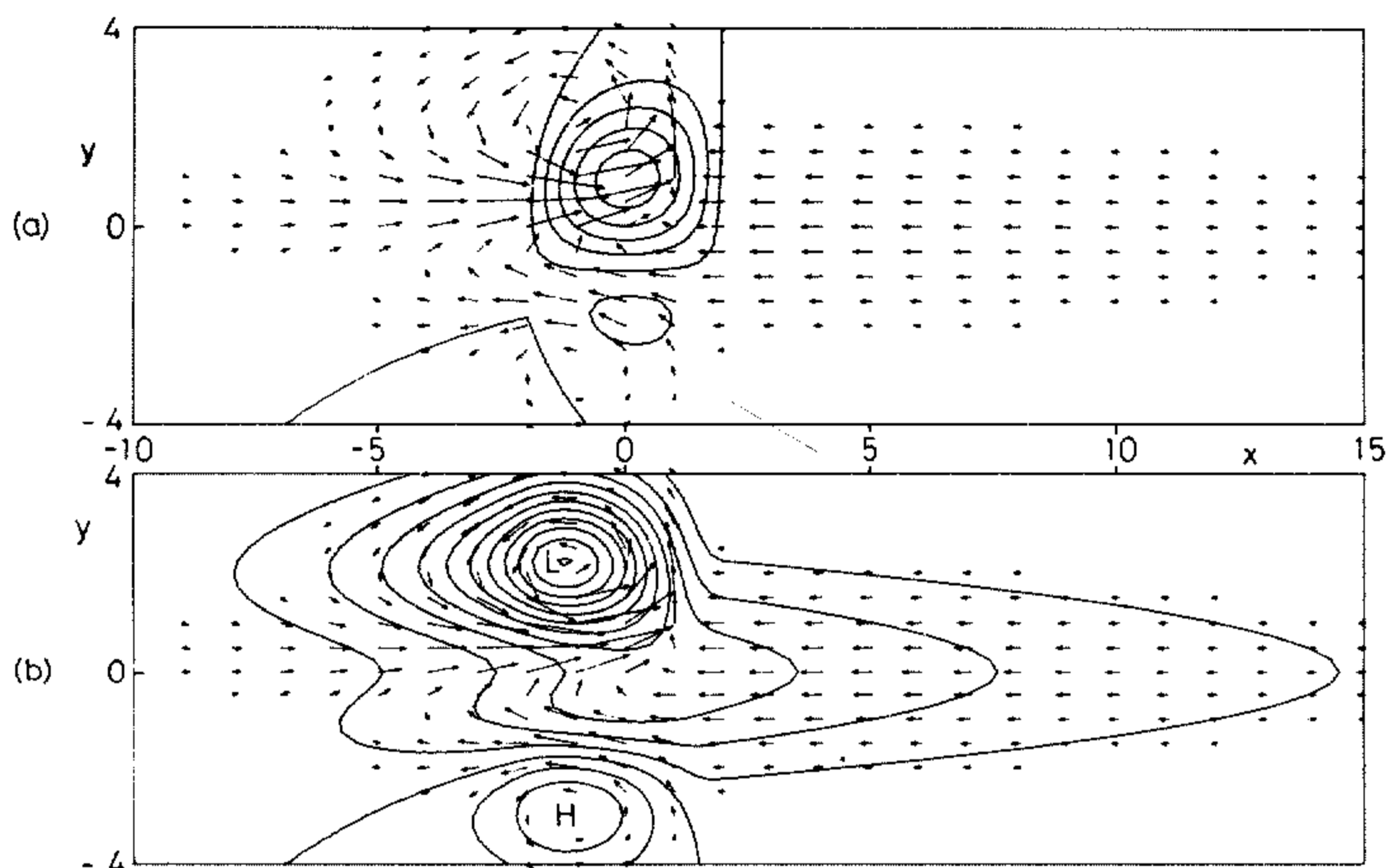
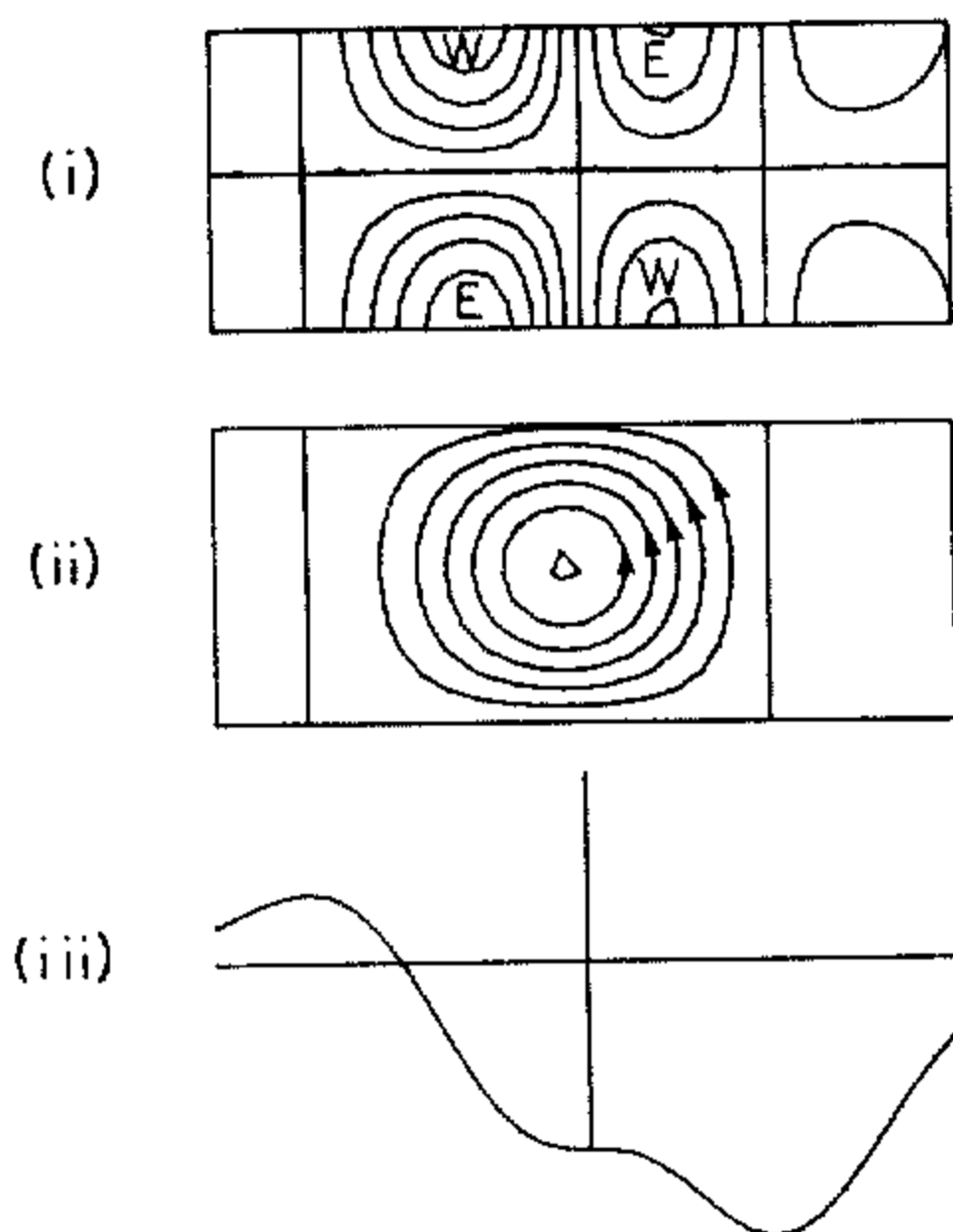


Figure 3. Solution obtained by adding the solutions shown in the two previous figures, corresponding to heating which is confined to the region  $|x| < 2$  and is mainly concentrated to the north of the equator. (a) Contours of vertical velocity  $w$  (contour interval 0.3) showing the dominance of the heating north of the equator. The flow to the east of the forcing region is the same as in Fig. 1, this being provided entirely by the symmetric part of the heating. West of the forcing, the westerly inflow is concentrated between the equator and  $y = 2$ . An easterly flow is found south of the equator. (b) Contours of perturbation pressure  $p$  (contour interval 0.3). The pattern is dominated by a low on the western flank of the heating region and by the equatorial trough. A high is found in the southern hemisphere.



(c) The zonally integrated solution showing (i) contours of eastward velocity, (ii) stream function contours and (iii) perturbation pressure. The dominant Hadley cell has nearly four times the circulation of the two cells shown in Fig. 1.

The meridionally integrated flow, being generated entirely by the symmetric part of the forcing, is the same as shown in Fig. 1(d). The circulation of the Hadley circulation shown in (c) is 70% of that in the Walker cell.

solutions for all the values of  $n$ . For forcing in a limited region, the  $Q$  part is the *only* part to give a response to the east. Also, as  $n$  increases, the decay rate to the west of long planetary waves increases as well, so that, with the exception of the forcing region and the area immediately to its west, the solution is largely that corresponding to the  $Q_0$  and  $Q_1$  forcing. For example, a solution corresponding to heating being mainly to the north of the equator is obtained simply by adding the solutions shown in Figs. 1 and 2. The result, shown in Fig. 3, agrees remarkably well with the lower tropospheric circulation for July, reproduced in Fig. 4, if the centre of the forcing region is taken at  $120^\circ\text{E}$ . Figure 4 also shows the

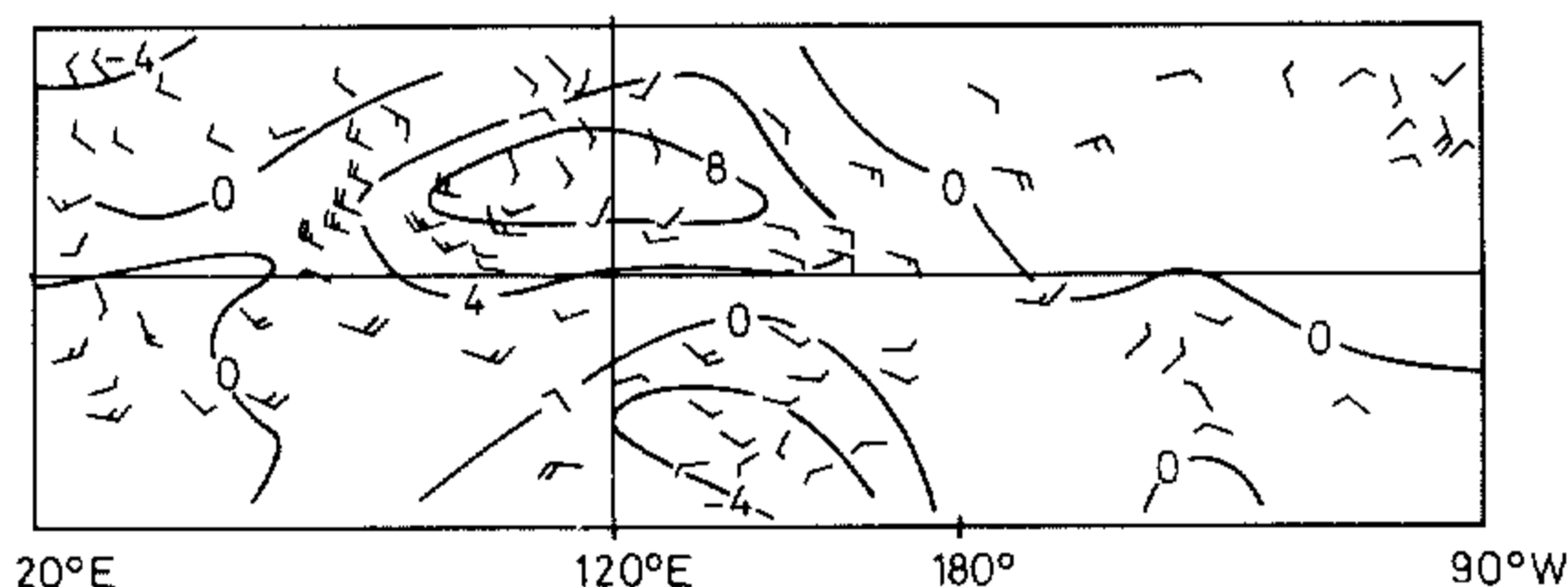


Figure 4. Contours of vertical velocity at 500 mb for the June–August season (units  $10^{-4} \text{ mbs}^{-1}$ , positive upwards) as deduced from the heat equation and presented in plate 9.1 of Newell *et al.* (1974). The contours are approximately those for the diabatic heating rate as the vertical velocity deduced this way is approximately equal to the heating rate divided by the temperature gradient. Also shown are 850 mb wind vectors after van de Boogaard (1977). (See e.g. Riehl 1979, Fig. 1.13.)

adiabatic heating rate, which is quite localized and has a fairly good resemblance to that used in the model.

The meridional circulation obtained in the model calculation (Fig. 3) is dominated by one cell as shown, and the meridionally averaged zonal circulation is precisely that shown in Fig. 1 (the 'Walker' circulation), since the asymmetric part gives no contribution. Contributions to the meridional circulation come only from the heating region and the area to the west. The *asymmetric* part of the heating gives the dominant contribution to this circulation in contradistinction to the Walker circulation which is driven entirely by the *symmetric* part of the heating.

It is of interest with respect to the circulation over the Indian Ocean to note that, although the forcing is mainly in the northern hemisphere, the response in the southern hemisphere is quite large. In particular, an easterly jet forms towards the west of the heating region in latitudes between  $y = -1$  and  $y = -2.5$ . This feeds a cross-equatorial flow into a strong westerly régime north of the equator.

It is also possible to obtain some idea of the effect of large topographic barriers like the East African Escarpment by adding a western boundary (across which no flow is allowed) to the model. The solution is modified by the reflection of planetary waves at the boundary and the flow in the western boundary layer itself can be calculated. This layer has a thickness of  $1/\varepsilon$  and a separate analysis is required to find its properties because the long-wave assumption (2.11) is not valid. A solution of the same forcing as in Fig. 3 was obtained in this way, but with a boundary at  $x = -5$ . The picture looks much the same as Fig. 3, but with pressure contours 'squashed inwards' a little by the boundary, and by the existence of a 'low-level jet' across the equator on the western boundary in order to complete the flow. The model of the jet is similar in concept to that of Anderson (1976) but the forcing is different and the friction formulation also differs.

Another topographic effect of importance is that of the Andes which presents a westerly surface flow from the Pacific into the heating zone over South America. To simulate that effect, a computer plot was obtained for the solution with the same forcing as in Fig. 1 but with a boundary at the western edge  $x = -2$  of the forcing region. The resulting flow looks almost the same as that shown in Fig. 1 for  $x > 0$ . On the western boundary, however, there are equatorward jets to complete the circulation and pressure contours are crowded together in order to give a pressure gradient which can balance this flow.

Another interesting exercise is to consider pure meridional circulations due to forcing which is independent of  $x$ . The solutions of (2.10) (omitting the  $\varepsilon^3$  term) are then functions of  $y$  only. For example, the intertropical convergence zone can be simulated by making  $Q$  a delta-function situated at a given latitude  $y_0$ . The solution can be given in terms of parabolic cylinder functions of order a half, which are tabulated in Abramowitz and Stegun (1965). Defining  $U$  as the cylinder function which decays as  $y \rightarrow \infty$ , i.e.

$$U(y) = \{\exp(-\frac{1}{2}y^2)\} \left(1 - y + \frac{1}{2} \frac{y^2}{2!} - \frac{3}{2} \frac{y^3}{3!} + \frac{15}{2} \frac{y^4}{4!} - \frac{37}{2} \frac{y^5}{5!} + \dots\right). \quad (6.1)$$

the appropriate solution, which must satisfy continuity of pressure at  $y = y_0$ , is given by

$$\begin{aligned} \frac{v}{\varepsilon} &= \frac{2u}{y} = U(y)/U'(y_0) \quad \text{for } y > y_0, \\ \text{and} &= -U(-y)/U'(-y_0) \quad \text{for } y < y_0, \\ p &= -U'(y)/U'(y_0) \quad \text{for } y > y_0, \\ \text{and} &= -U'(-y)/U'(-y_0) \quad \text{for } y < y_0. \end{aligned} \quad (6.2)$$

This solution features a trough at the singularity and all the rising motion is located over the trough. The greater part of the low-level inflow into the convergence zone comes from the equatorial side with a smaller inflow from the poleward side. Figure 5 shows the velocity field obtained by adding the solution (6.2) to that shown in Fig. 1 with the value of  $y_0$  equal to unity. One noticeable effect is the splitting of the easterlies in the region  $x > 2$  into a northern branch and a southern branch.

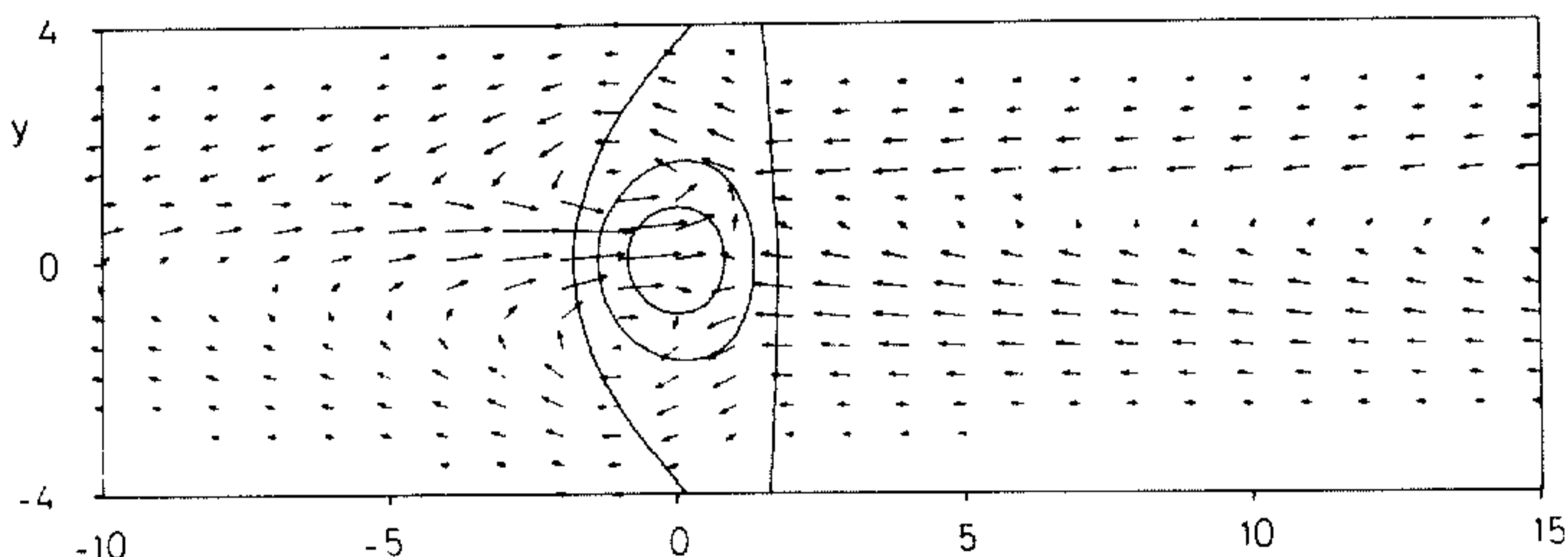


Figure 5. Flow pattern obtained by adding to the symmetric solution the effect of a line source of heat (Model ITCZ) at  $y=1$ . The contours show the vertical velocity due to the symmetric part of the forcing, but there is also a concentrated upflow along the line source. One effect of the model ITCZ is to split the easterlies into two jets, with relatively weak flow between the equator and the convergence line.

## 7. DISCUSSION

The idea for this paper arose from trying to interpret the equatorial part of the flow in the lower troposphere shown in Fig. 1(c) of Krueger and Winston (1974) in the light of the authors' comment that the principal sources of heating are located in the land areas of the equatorial zone. That led to the idea that Kelvin waves emanating from the Indonesian region would create an extensive region of easterlies to the east (which in fact extend over the whole of the Pacific) while planetary waves would create a more limited region (roughly one-third the size) of westerlies to the west (they in fact extend over the whole of the Indian ocean, but tend to be confined to one hemisphere (Lettau 1973)). A solution was quickly sketched on the basis of a simplified version of the above ( $F(x)$  constant in  $|x| < 2$  and zero outside, terms in  $\varepsilon$  ignored in the forcing region) and gave a picture quite similar to the observed one, including the unexpected poleward outflow from the heating region.

Subsequent further reading of the above-mentioned paper drew attention to the paper by Webster (1972), which included a numerical solution of a two-layer model with perturbations to a mean flow like that observed and with a heating distribution rather similar to that considered here. The results (shown in his Fig. 9) are very similar to those displayed in Fig. 1(b) above, and Webster correctly interpreted the flow to the east of the forcing region in terms of a Kelvin wave. However, the other features of the flow are not so readily interpreted in the numerical solution, and it is felt that the above solutions are useful because of their simple form and the ease with which they can be interpreted. In fact, they give an illuminating picture of the tropic circulation, yet only involve the solution of simple, ordinary differential equations.

Although the solutions are obtained for one mode only, they can in principle be super-

posed in the manner described at the beginning of section 2 to give solutions for a semi-infinite atmosphere. Solutions for transient problems constructed this way satisfy the correct radiation condition at  $z = \infty$ , and the integrals can be evaluated numerically. Such a construction, however, goes rather beyond the aim of this paper, which is to illustrate important properties of the response by means of simple analytic solutions.

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